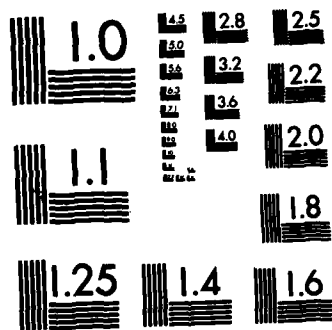END
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A CHARACTERIZATION OF BINARY COMPLETELY SYMMETRIC CONNECTED BLOCK DESIGNS WITH $v = b$ BASED ON $n \equiv 0 (\text{MOD } v)$ EXPERIMENTAL UNITS

by

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ABSTRACT

It is shown that among binary and connected block designs with and without equal block sizes and equal replications only a symmetric balanced incomplete block design produces a completely symmetric information matrix for the treatment effects whenever the number of blocks is equal to the number of treatments and the number of experimental units is a multiple of the number of treatments. This result contains a known result as a special case.

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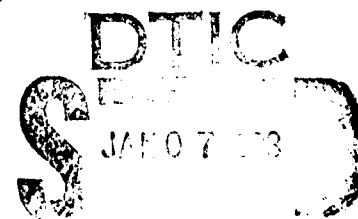**20. ABSTRACT (Continue on reverse side if necessary and identify by block number)**

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1. INTRODUCTION

Suppose v treatments are to be compared via n experimental units grouped into b blocks with k_j experimental units in block j. An arrangement of the v treatments into the n experimental units is called a _design_. Let D be the collection of all such designs and let r_{d1} be the number of experimental units which design d assigns to treatment i. We associate the following matrix

$$C_d = R_d - N_d K^{-1} N_d'$$

with design d, where R_d and K_d stand for diagonal matrices with diagonal elements $r_{d1}, r_{d2}, \ldots, r_{dv}$ and $k_1, k_2, \ldots, k_b$ respectively and $N_d = (n_{dij})$, represents the v x b incidence matrix of the treatments versus blocks. Throughout the paper we shall adopt the following standard notation. A' and Tr A for the transpose and the trace of the matrix A respectively; I for the identity matrix and J for a matrix with all entries equal to one. The dimensions of the matrices will be clear from the context.

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The matrix C_d plays an important role in the statistical analysis of the data collected under design d whenever the model of response is the usual homoscedastic linear additive model which specifies that the expectation of an observation on treatment i in block j is $\mu + \alpha_i + \beta_j$ (general mean+treatment effect+block effect), where μ, α_i and β_j are unknown constants, $1 \leq i \leq v$, $1 \leq j \leq b$ with the additional assumption that the n observations are uncorrelated with common variance σ^2. Under this model the reduced normal equations for estimating the vector of treatment effects can be written as $C_d \hat{\alpha} = Q_d$, where $Q_d = (T_{d1}, T_{d2}, \ldots, T_{dv})' - N_d K^{-1} (B_{d1}, B_{d2}, \ldots, B_{db})'$. Here T_{di} stands for the total of all observations obtained under treatment i and B_{dj} stands for the total of all observations produced in block j. We recall that if $q'\hat{\alpha}$ is an estimable linear contrast in treatment effects then under design d the BLU estimator of $q'\alpha$ is $q'\hat{\alpha} = q'C_d^- Q_d$ with variance $q'\hat{\alpha} = q'C_d^- q\sigma^2$. Here A^- is a generalized inverse of A, that is, any matrix A^- as long as $AA^- A = A$. For this reason C_d is known as the information matrix of the estimable treatment contrasts.

As we see from the definition of C_d each d uniquely determines its associated C_d matrix. Fortunately, in general the converse is not true, that is, there are different designs which yield identical information matrices. Therefore, in such a situation the statistician has more than one design at his disposal for his client who insists on running a design with a certain form

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for the information matrix of his design. One particular pattern which yields a great deal of optimality is complete symmetry by which we mean C_d to be of the form $a_1 I + a_2 J$. Kiefer (1975) proved that any design which yields a completely symmetric C_d matrix is universally optimal if $\operatorname{Tr} C_d$ is maximum in the class of competing designs. In this paper we shall prove that if $b = v$ and $n \equiv 0 \pmod{v}$ then C_d will be completely symmetric if and only if d is a symmetric BIB design. The sufficiency of the condition is obvious. To prove its necessity we need to recall some basic old results and prove some new ones which we shall do in Section 2. We would like to point out that our result contains the interesting result of Rao (1966), who proved that an equireplicate binary balanced design with $b = v$ is a symmetric BIB design. Specifically, we shall show that to obtain the conclusion of Rao (1966) we can start with a much weaker assumption. Namely, rather than assuming that d is equally replicated we only have to assume that the number of available experimental units, n, is a multiple of v. Of course, somewhere in the process of the proof we will see that, under the given conditions, our assumptions and the assumptions of Rao (1966) are indeed equivalent. But a priori we did not have to start with the strong condition of equal replications. A result has to be established (Proposition 2.6) in order to enable us to weaken the a priori condition or to enlarge the class of competing designs.

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2. CHARACTERIZATION

In the sequel for simplicity we shall drop the subscript d to the matrices and quantities associated with design d. In Propositions 2.1-2.6 we establish some results needed to prove our result stated in Theorem 2.1 which was given originally in Hedayat (1972). The stated results can be easily verified whenever the proof is omitted.

Proposition 2.1. If d is a completely symmetric connected design with

$$C = a_1 I + a_2 J$$

then,

$$a_1 = TrC/(v-1) \quad \text{and} \quad a_2 = -TrC/v(v-1).$$

Proposition 2.2. If d is a binary completely symmetric connected design with

$$C = a_1 I + a_2 J$$

then,

$$a_1 = (n-b)/(v-1) \quad \text{and} \quad a_2 = -(n-b)/v(v-1).$$

Definition 2.1. d is said to be a proper design if $k_j = k$.

Since for a proper design $n = bk$ then we have

Proposition 2.3. If d is a binary, proper and completely symmetric connected design, then

$$C = \frac{b(k-1)}{v-1} I - \frac{b(k-1)}{v(v-1)} J.$$

Given that d is completely symmetric, then in some settings we can make useful inferences about R and K.

Proposition 2.4. If d is a binary, proper and completely symmetric connected design, then d is equireplicated.

Proof: $C_{ii} = r_i (1 - \frac{1}{k})$ since d is binary and proper. Also, $C_{ii} = (n-b)/v$ since d is completely symmetric. Thus, $r_i = n/v$.

Proposition 2.5. If d is a binary, proper, equireplicated and completely symmetric connected design, then d is a BIB design.

Proof: On one hand

$$C_{ii'} = -1(k) \sum_{j=1}^{b} n_{ij} n_{i'j}.$$

One the other hand

$$C_{ii'} = -(n-b)/v(v-1),$$

and hence

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$$\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} \, n_{i'j} = r(k-1)/(v-1).$$

Proposition 2.6. If d is a binary and completely symmetric connected design with $b = v$ then d is equireplicated if $n \equiv 0 \pmod{v}$.

Proof: Since d is binary and completely symmetric, then

(1) $\quad C_{ii} = (n-b)/v = (n-v)/v = (n/v) - 1.$

On the other hand,

(2) $\quad C_{ii} = r_i - \sum_{j=1}^{b} \dfrac{n_{ij}}{k_j}.$

(1) and (2) imply that

$$r_i = r - 1 + \sum_{j=1}^{b} \frac{n_{ij}}{k_j} \quad \text{where} \quad r = n/v.$$

Since r_i is an integer, this implies that $\sum_{j=1}^{b} \dfrac{n_{ij}}{k_j}$ is an integer greater than or equal to one. However, we observce that

$$\sum_{i=1}^{v} \left\lceil \sum_{j=1}^{b} \frac{n_{ij}}{k_j} \right\rceil = \sum_{j=1}^{b} \frac{1}{k_j} \sum_{i=1}^{v} \frac{1}{k_j} (k_j) = b = v,$$

that is, $v \; (=b)$ terms ≥ 1 add up to v which implies that each term $= 1$. Therefore, $r_i = r - 1 + 1 = r;$ $i = 1,2,\ldots,v.$

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__Theorem__ __2.1__. If d is a binary and completely symmetric connected design with b = v and n ≡ 0(mod v), then d is a symmetric BIB design.

__Proof__: By Proposition 2.6, it is sufficient to prove that d is proper.

$$C = R - NK^{-1}N' = rI - NK^{-1}N', \qquad n = vr$$

$$= \frac{n-b}{v-1} I - \frac{n-b}{v(v-1)} J$$

$$= \frac{v(r-1)}{v-1} I - \frac{r-1}{v-1} J .$$

Thus

$$NK^{-1}N' = rI - \frac{v(r-1)}{v-1} I + \frac{r-1}{v-1} J$$

$$= \frac{v-r}{v-1} I + \frac{r-1}{v-1} J .$$

Since

$$\left\lceil \frac{v-r}{v-1} I + \frac{r-1}{v-1} J \right\rceil^{-1} = \frac{v-1}{v-r} I - \frac{r-1}{r(v-r)} J$$

we have

$$[NK^{-1}N']^{-1} = (N')^{-1} KN^{-1} = \frac{v-1}{v-r} I - \frac{r-1}{r(v-r)} J.$$

By premultiplying by N' and pastmultiplying by N the both sides of the above relation we obtain

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$$K = \frac{v-1}{v-r} N'N \; - \; \frac{r-1}{r(v-r)} N'JN.$$

Note that $J = 11'$ and $N'1 = (k_1, \ldots, k_b)$. Since K is a diagonal matrix with the ith diagonal entry equal to k_i we thus observe that:

(3) $\quad k_i = k_i(v-1)/(v-r) \; - \; k_i^2(r-1)/r(v-r),$

$$i, j = 1, 2, \ldots, b; \; i \neq j.$$

(4) $\quad 0 = \lambda_{ij}(v-1)/(v-r) \; - \; k_i k_j(r-1)/r(v-r)$

Because $k_i > 0$, we conclude from (3) that $k_i = r$. Consequently, (4) implies that $\lambda_{ij} = r(r-1)/(v-1)$. This completes the proof.

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